ON TRANSFORMATION OF COVARIANCE MATRICES BETWEEN LOCAL CARTESIAN COORDINATE SYSTEMS AND COMMUTATIVE DIAGRAMS

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ABSTRACT

Transformations of covariance matrices between several local Cartesian coordinate systems (WGS72, spherical, geodetic) are obtained by simply using the rotation matrices relating any two frames. The approach followed here combines all possible rotations through a general commutative diagram, hence departing from the conventional evaluation of one Jacobian matrix for each functional relationship between two sets of coordinates, a task sometimes cumbersome or difficult to accomplish.

1. INTRODUCTION.

By definition the covariance matrix Σ_X of a n-dimensional vector random variable X can be written [e.g. consult Uotila, 1970; Vaníček and Krakiwsky, 1980, p.197; Leick, 1980; Meissl, 1982]

$$\Sigma_X = E[\{X-E(X)\}\{X-E(X)\}^t] \tag{1.1}$$

where E stands for the statistical expectation operator and t for transpose. Restricting the present discussion to the three-dimensional space of our ordinary experience, the coordinates of any point P can be mathematically represented in any of the following vector-matrix forms

$$X = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \{x\} \tag{1.2}$$

and the associated symmetric covariance matrix of the position by

$$\Sigma_X = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ & \sigma_y^2 & \sigma_{yz} \\ \text{sym} & & \sigma_z^2 \end{bmatrix} \tag{1.3}$$

Equation (1.3) gives explicitly the covariance matrix (also termed by some authors variance-covariance matrix) of the coordinates of the point referred to a prescribed Cartesian system with arbitrary origin. For instance, when reducing observations to satellites in the GPS (Global Positioning System) constellation, the three-dimensional relative position (in the WGS72 system as defined by the satellite ephemerides) of any station P with respect to a base station A together with the corresponding covariance matrix Σ_X at P are determined. Notice that the diagonal elements of Σ_X are the variances or "mean square errors" of the coordinates of P along the x,y and z axes. The "root mean square errors" (rms) or standard deviations are denoted as usual by σ_x, σ_y and σ_z.

Readers unfamiliar with the GPS technology and methods may find a comprehensive account emphasizing the practical aspects of GPS surveying in [Hothem et al., 1984].

The question addressed in this paper is general and relates to the transformation of covariance matrices Σ_X between different local Cartesian coordinate systems commonly used in surveying and geodesy.

2. TRANSFORMATION OF Σ_X BETWEEN CARTESIAN COORDINATE SYSTEMS.

It is known that if we want to express Σ_X with respect to another coordinate system having the same origin but different orientation, only a rotation will be involved. For simplicity possible scale differences along the axes of the two coordinate systems are neglected. Thus if the final rotated coordinates are expressed by

$$(2.1) \qquad \tilde{X} = \begin{Bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{Bmatrix} = \{\tilde{x}\}$$

then clearly

$$(2.2) \qquad \tilde{X} = RX \qquad \text{or} \qquad \{\tilde{x}\} = R\{x\} \; .$$

This matrix transformation can also be represented symbolically as a mapping between the two coordinate systems denoted here in parentheses, namely

$$(2.3) \qquad \begin{array}{ccc} (x,y,z) & \xrightarrow{R} & (\tilde{x},\tilde{y},\tilde{z}) \\ \text{initial} & & \text{final} \\ \text{rotating} & & \text{fixed} \\ \text{first} & & \text{second} \end{array}$$

or equivalently as a mapping between the two sets of coordinates

$$(2.4) \qquad \{x\} \xrightarrow{R} \{\tilde{x}\} \quad .$$

R is a proper orthogonal (rotation) matrix (i.e. $R^t = R^{-1}$ and $|R| = +1$) that can be parameterized as a function of the angles involved in the rotation of the frame (x,y,z) to a position parallel or in coincidence with the $(\tilde{x},\tilde{y},\tilde{z})$ frame. Later it will be shown how the different rotation matrices R are actually obtained in the particular cases involving transformations between local spherical and geodetic frames and the previously mentioned reference frame WGS72.

Let us return to the primary subject of this paper, how to determine the covariance matrix $\Sigma_{\tilde{X}}$ with respect to the rotated frame $(\tilde{x},\tilde{y},\tilde{z})$ when Σ_X and the rotation R are given. By analogy with equation (1.1) we can state that the transformed covariance must have the form

$$(2.5) \qquad \Sigma_{\tilde{X}} = E[\{\tilde{X}-E(\tilde{X})\}\{\tilde{X}-E(\tilde{X})\}^t] \quad .$$

Substituting equation (2.2) above and applying the properties of the expectation, in particular

$$(2.6) \qquad E(RX) = R\,E(X)$$

it immediately follows that

$$(2.7) \qquad \Sigma_{\tilde{X}} = R\,E[\{X-E(X)\}\{X-E(X)\}^t]\,R^t$$

and after replacing the basic definition (1.1), finally

$$(2.8) \qquad \Sigma_{\tilde{X}} = R\,\Sigma_X\,R^t \quad .$$

Therefore, in order to compute the covariance matrix $\Sigma_{\tilde{X}}$ with respect to a new rotated coordinate system $\tilde{X}$, knowing the original covariance matrix Σ_X and the rotation matrix R of the transformation, it is necessary only to multiply the original covariance matrix from the left by R and from the right by R^t, or vice versa. Equation (2.8) is general and applies to any Cartesian covariance matrix transformation. The term "Cartesian" implies that all the units of the elements in the covariance matrix have the same dimension, in this case, linear units

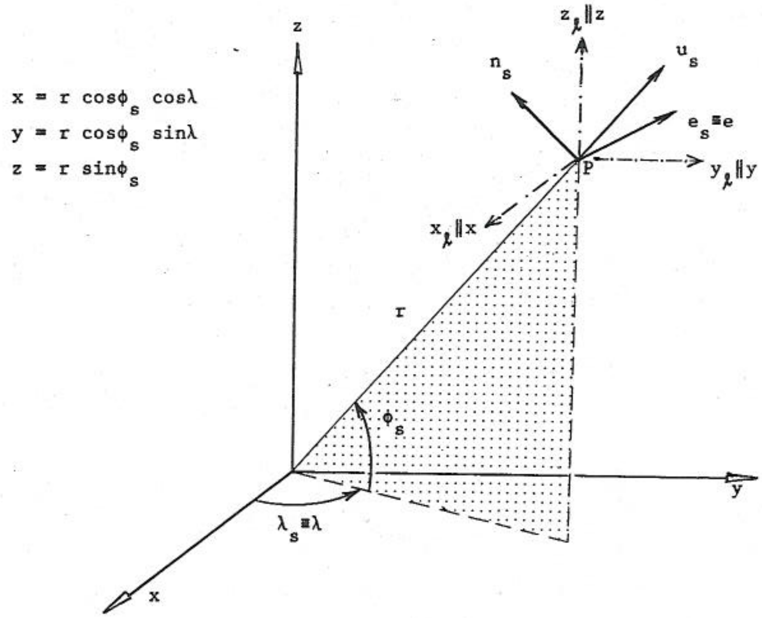

Fig 1. Local Cartesian and spherical coordinate systems at P

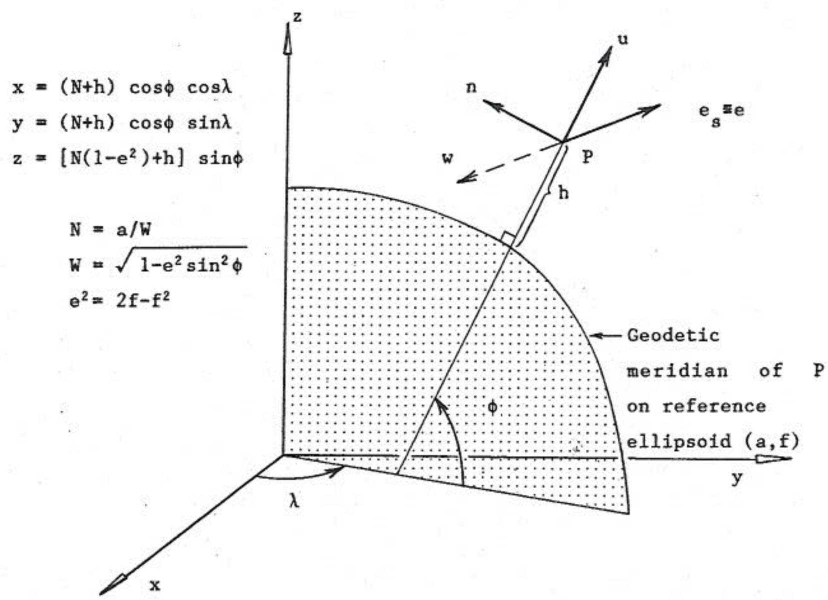

Fig 2. Local geodetic coordinate systems at P

squared (e.g. m^2, cm^2, mm^2).

Obviously equation (2.8) represents the so called "law of propagation of covariance" in the particular case in which the functional relationship between the two random variables $\tilde{X}$ and X is linear, and consequently the conventional Jacobian matrix $J = \partial\tilde{X}/\partial X$ reduces to R. Incidentally, readers familiar with tensor calculus would have immediately recognized equation (2.8) as the standard transformation under rotation of three dimensional second-rank Cartesian tensors. Therefore as a corollary it can be stated that covariance matrices are second-rank tensors.

3. ROTATION MATRICES BETWEEN LOCAL SPHERICAL AND GEODETIC CARTESIAN SYSTEMS.

Assume a point P in space. It is always possible to define a local Cartesian coordinate system with origin at P which is parallel to the geocentric WGS72. Naturally there are other possible choices of local coordinate systems with the same P origin. There is some disagreement in the geodetic literature about the notation used for these local frames and the selection of right- over left-handed systems [e.g. Molodenskii et al., 1960, p.14; Rapp, 1984]. In this presentation only right-handed coordinate systems will be considered. To avoid any possible confusion, the following notation and terminology will be adopted :

a) Local spherical coordinate system, (e_s, n_s, u_s).[Refer to Fig.1]

origin: The point P as defined by the geocentric Cartesian coordinates x,y,z or the curvilinear spherical coordinates λ_s, ϕ_s, r, where λ_s = spherical longitude; ϕ_s = spherical (geocentric) latitude and r = radius vector from the geocenter to P.

u_s axis: Normal through P to the sphere of radius r. Positive in the outward (<u>up</u>)direction.

n_s axis: Normal to u_s and tangent to the meridian through P. Positive <u>north</u>, the direction of increasing ϕ_s.

e_s axis: Normal to u_s and n_s at P forming a right-handed orthogonal triad. Positive <u>east</u>, the direction of increasing λ_s.

b) Local geodetic coordinate system, (e,n,u). [Refer to Fig.2].

origin: The point P defined as above by the geocentric coordinates x,y,z or the geodetic coordinates λ, ϕ, h, where now λ = geodetic longitude, ϕ = geodetic latitude, h = geodetic height (h>0 for points above the reference ellipsoid). Notice that $\lambda_s = \lambda$ due to the rotational symmetry of the ellipsoid.

u axis : Normal through P to the reference ellipsoid. Positive in the outward (up) direction.

e axis : Normal to u and the geodetic meridian plane of P (when h=0 tangent to the geodetic parallel of P). Positive east, the direction of increasing λ.

n axis : Perpendicular to e and u forming a right-handed triad (when h=0 tangent to the geodetic meridian of P). Positive north, the direction of increasing ϕ.

In order to obtain the rotation matrix R_s of the transformation (mapping) between a local frame parallel to WGS72 denoted (x_ℓ, y_ℓ, z_ℓ) and the local spherical frame (e_s, n_s, u_s) at P, namely

$$(3.1) \qquad \underset{\text{local WGS72}}{(x_\ell, y_\ell, z_\ell)} \xrightarrow{R_s} \underset{\text{local spherical}}{(e_s, n_s, u_s)}$$

it is necessary to rotate the (x_ℓ, y_ℓ, z_ℓ) coordinate system to a final position which is coincident with the local (e_s, n_s, u_s). Alternatively, this can be easily visualized by instead rotating the geocentric WGS72 frame about the third (z) and first (x) coordinate axes by angles $\lambda+\frac{1}{2}\pi$ and $\frac{1}{2}\pi-\phi_s$ respectively (see Fig.1), until achieving parallelism with the (e_s, n_s, u_s) frame. Consequently

$$(3.2) \qquad R_s = R_1(\tfrac{1}{2}\pi-\phi_s)\, R_3(\lambda+\tfrac{1}{2}\pi)$$

$$= \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi_s\cos\lambda & -\sin\phi_s\sin\lambda & \cos\phi_s \\ \cos\phi_s\cos\lambda & \cos\phi_s\sin\lambda & \sin\phi_s \end{bmatrix}$$

where in general the individual rotations about the first and third axes are written [see Goldstein, 1950, p.109; Hotine, 1969, p.72; Mueller, 1975, p.80]

$$(3.3) \qquad R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$(3.4)\qquad R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These matrices are consistent with counterclockwise positive rotations about the axes of right-handed coordinate systems. A convenient condensed algorithm very useful to code the three fundamental rotation matrices $R_i(\theta)$, $i=1,2,3$ is given in [Kaula; 1966, p.13]. The quaternion formulation corresponding to these three basic rotations was discussed in [Pope, 1970].

For completeness the following properties of rotation matrices should be recalled

$$(3.5)\qquad R_i^{-1}(\theta) = R_i^t(\theta) = R_i(-\theta)$$

$$(3.6)\qquad R_i(\theta)\ R_i(\omega) = R_i(\theta+\omega)$$

$$(3.7)\qquad [R_i(\theta)\ R_j(\omega)]^t = R_j^t(\omega)\ R_i^t(\theta) = R_j(-\omega)\ R_i(-\theta)$$

As usual the product of rotations in equation (3.2) operates from right to left. The matrix of the first rotation (transformation) performed is written to the right, while each successive rotation afterwards operates to its left in sequential order. Later in sections four and five this fundamental property will be applied constantly when "commutative diagrams" are introduced.

Now let us define the rotation matrix R of the transformation between the local WGS72 and local geodetic coordinate systems, essentially the mapping

$$(3.8)\qquad \underset{\text{WGS72}}{(x_\ell, y_\ell, z_\ell)} \xrightarrow{\ R\ } \underset{\text{local geodetic}}{(e,n,u)}$$

It can be proved easily in a way similar to (3.2) that

$$(3.9)\qquad R = R_1(\tfrac{1}{2}\pi-\phi)\ R_3(\lambda+\tfrac{1}{2}\pi)$$

Note that the explicit matrix form of the above equation is exactly (3.2) after replacing ϕ_s by ϕ.

In conclusion, the final transformations between a known covariance matrix in the WGS72 system and the local spherical or geodetic coordinate systems can be written respectively by

$$\Sigma_{(e_s,n_s,u_s)} = R_s \, \Sigma_{WGS72} \, R_s^t \tag{3.10}$$

and

$$\Sigma_{(e,n,u)} = R \, \Sigma_{WGS72} \, R^t \tag{3.11}$$

where the following standard notation is implied

$$\Sigma_{(e,n,u)} = \begin{bmatrix} \sigma_e^2 & \sigma_{en} & \sigma_{eu} \\ & \sigma_n^2 & \sigma_{nu} \\ \text{sym} & & \sigma_u^2 \end{bmatrix} \tag{3.12}$$

It should be stressed at this point that covariance and correlation matrices do not transform according to the same rules. For example, defining the symmetric correlation matrix of the coordinates of point P in the WGS72 system by

$$[\rho_{WGS72}] = \begin{bmatrix} 1 & \rho_{xy} & \rho_{xz} \\ & 1 & \rho_{yz} \\ \text{sym} & & 1 \end{bmatrix} \tag{3.13}$$

where as usual the correlation coefficient between any two variables $(i,j = x,y,z)$ is expressed by

$$\rho_{ij} = \sigma_{ij}/\sigma_i\sigma_j \tag{3.14}$$

then, it can be proved that in general

$$[\rho_{(e,n,u)}] \neq R \, [\rho_{WGS72}] \, R^t \tag{3.15}$$

and consequently, correlation matrices are not tensors.

The preceding result suggests that it may be more useful to know the covariance matrix rather than the correlation matrix, since this expedites the immediate application of transformations such as (3.10) and (3.11) from which the correlation coefficients, if necessary, can be computed. Nevertheless because the magnitude of the correlations is important for understanding at first glance certain characteristics of the variables involved, as an alternative the full non-symmetric covariance-correlation matrix can be given. In the notation of this paper we will write

$$(3.16) \qquad \sum_{\rho}{}_{WGS72} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \rho_{xy} & \sigma_y^2 & \sigma_{yz} \\ \rho_{xz} & \rho_{yz} & \sigma_z^2 \end{bmatrix} .$$

It has been assumed throughout this work that east longitudes (spherical or geodetic) are positive (see also Fgs. 1 and 2). Howewer sometimes it is preferable to have west longitudes positive; we will now consider the transformations required in order to change from one of these frames to another. A common right-handed coordinate system with positive W-longitudes is defined by the triad (n,w,u) where n=north, w=west and u=up.

It is immediately evident that the transformation between the local geodetic frames with E- and W-longitude positive respectively, can be expressed by the mapping

$$(3.17) \qquad (e,n,u) \underset{R_3^t(\frac{1}{2}\pi)}{\overset{R_3(\frac{1}{2}\pi)}{\rightleftarrows}} (n,w,u)$$

where after applying (3.4), we can write explicitly

$$(3.18) \qquad R_3(\tfrac{1}{2}\pi) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and hence the relation between the covariance matrices corresponding to the mapping (3.17) transforms according to

$$\Sigma_{(n,w,u)} = R_3(\tfrac{1}{2}\pi)\ \Sigma_{(e,n,u)}\ R_3^t(\tfrac{1}{2}\pi) \tag{3.19}$$

or similarly

$$\Sigma_{(e,n,u)} = R_3^t(\tfrac{1}{2}\pi)\ \Sigma_{(n,w,u)}\ R_3(\tfrac{1}{2}\pi) \tag{3.20}$$

Incidentally because in this case the axes of the two coordinate frames are aligned (although not coincident), the correlation matrices transform in a way similar to equations (3.19) and (3.20); for example

$$[\rho_{(n,w,u)}] = R_3(\tfrac{1}{2}\pi)\ [\rho_{(e,n,u)}]\ R_3^t(\tfrac{1}{2}\pi) \tag{3.21}$$

4. THE COMMUTATIVE DIAGRAM CONCEPT.

It is possible to establish a general visual relationship between all different coordinate systems described before by the schematic representation of a "commutative diagram". A commutative diagram is a symbolic way of showing at once all available transformations connecting different sets of coordinates or the coordinate systems to which they refer. Essentially it is a generalization of the basic mapping (2.3) when more than two coordinate systems are involved. The word "commutative" when referring to diagrams implies the existence of the inverse transformation for every individual mapping appearing in the diagram. This condition is always fulfilled when treating rotations because of the property of non-singularity and the relation $R^{-1} = R^t$.

For example, in our particular case combining the mappings (3.1) and (3.8) we can establish the commutative diagram relating the three local coordinate systems described previously as follows

$$\begin{tikzcd}
(e_s, n_s, u_s) \arrow[rr, shift left, "\mathbf{R}"] \arrow[dr, shift left, "R_s^t"] & & (e, n, u) \arrow[ll, shift left, "\mathbf{R}^t"] \arrow[dl, shift left, "R^t"] \\
& (x_\ell, y_\ell, z_\ell) \arrow[ul, shift left, "R_s"] \arrow[ur, shift left, "R"] &
\end{tikzcd} \tag{4.1}$$

Although the values of the matrices R_s and R appearing in the commutative diagram (4.1) were given above by equations (3.2) and (3.9) respectively, the matrix $\mathcal{R}$ nevertheless remains to be defined. As clearly shown in the diagram the rotation $\mathcal{R}$ transforms the local spherical into the local geodetic coordinate system. This mapping can be obtained by using any different known path that goes from (e_s, n_s, u_s) to (e,n,u) and applying from right to left every rotation matrix encountered in the circuit. Therefore

$$(4.2) \qquad \mathcal{R} = R\, R_s^t$$

Alternatively, if we want to transform coordinates from the geodetic to the spherical system, the inverse transformation should be used, that is

$$(4.3) \qquad \mathcal{R}^{-1} = \mathcal{R}^t = R_s\, R^t$$

5. THE GENERAL COMMUTATIVE DIAGRAM RELATING LOCAL CARTESIAN SYSTEMS

By expanding the number of local Cartesian systems in the commutative diagram to include the E-positive as well as W-positive cases, a more general diagram may be established. This is shown explicitly below .

$$(5.1) \qquad \begin{tikzcd}[column sep=large, row sep=large]
(e_s, n_s, u_s) \arrow[rr, shift left, "\mathcal{R}"] \arrow[dr, shift left, "R_s^t"] \arrow[dd, shift left, "R_3(\frac{1}{2}\pi)"] & & (e,\ n,\ u) \arrow[ll, shift left, "\mathcal{R}^t"] \arrow[dl, shift left, "R^t"] \arrow[dd, shift left, "R_3(\frac{1}{2}\pi)"] \\
& (x_\ell, y_\ell, z_\ell) \arrow[ul, shift left, "R_s"] \arrow[ur, shift left, "R"] \arrow[dl, shift left, "\mathbf{R}_s"] \arrow[dr, shift left, "\mathbf{R}"] & \\
(n_s, w_s, u_s) \arrow[uu, shift left, "R_3^t(\frac{1}{2}\pi)"] \arrow[ur, shift left, "\mathbf{R}_s^t"] \arrow[rr, shift left, "\sigma R"] & & (n,\ w,\ u) \arrow[uu, shift left, "R_3^t(\frac{1}{2}\pi)"] \arrow[ul, shift left, "\mathbf{R}^t"] \arrow[ll, shift left, "\sigma R^t"]
\end{tikzcd}$$

Applying the conventional rules described before, and using different paths, as an illustration we may write the transformation between local spherical W-longitude positive and local geodetic E-longitude positive as the mapping

$$(5.2) \qquad (n_s, w_s, u_s) \underset{\mathit{R}^t}{\overset{\mathit{R}}{\rightleftarrows}} (e, n, u)$$

where R can be defined in any of the following ways

$$(5.3) \qquad \begin{aligned} \mathit{R} &= \mathrm{R}\,\mathrm{R}_s^t\,\mathrm{R}_3^t(\tfrac{1}{2}\pi) = \mathfrak{R}\,\mathrm{R}_3^t(\tfrac{1}{2}\pi) = \mathrm{R}\,\mathbf{R}_s^t \\ &= \mathfrak{R}\,\mathrm{R}_s\,\mathbf{R}_s^t = \mathrm{R}_3^t(\tfrac{1}{2}\pi)\,\mathbf{R}\,\mathbf{R}_s^t = \mathrm{R}_3^t(\tfrac{1}{2}\pi)\,{}^{\sigma}\mathrm{R} \quad . \end{aligned}$$

Notice that all of these transformations will give the same final rotation matrix R. Therefore through a commutative diagram it is possible to see immediately which path will be the most appropriate in order to compute any desired transformation matrix as a function of the known ones. Recall that in this particular case we have "a priori" R, R_s, and $\mathrm{R}_3(\frac{1}{2}\pi)$ or their transposes. The matrix $\mathfrak{R}$ was given in (4.2), thus we can compute $\mathbf{R}$, $\mathbf{R}_s$ and ${}^{\sigma}\mathrm{R}$ if desired

$$(5.4) \qquad \mathbf{R} = \mathrm{R}_3(\tfrac{1}{2}\pi)\,\mathrm{R}$$

$$(5.5) \qquad \mathbf{R}_s = \mathrm{R}_3(\tfrac{1}{2}\pi)\,\mathrm{R}_s$$

and

$$(5.6) \qquad {}^{\sigma}\mathrm{R} = \mathbf{R}\,\mathbf{R}_s^t$$

Nevertheless it should be noticed that it is possible to obtain the rotation matrices of the transformations between any two sets of coordinates in a commutative diagram only as a function of the initially known rotations, in this example

$$(5.7) \qquad \begin{Bmatrix} e \\ n \\ u \end{Bmatrix} = \mathrm{R}\,\mathrm{R}_s^t\,\mathrm{R}_3^t(\tfrac{1}{2}\pi) \begin{Bmatrix} n_s \\ w_s \\ u_s \end{Bmatrix} = \mathit{R} \begin{Bmatrix} n_s \\ w_s \\ u_s \end{Bmatrix}$$

and the corresponding covariance matrix is written

$$(5.8) \qquad \Sigma_{(e,n,u)} = \mathit{R}\,\Sigma_{(n_s,w_s,u_s)}\mathit{R}^t \quad .$$

An extension of the use of commutative diagrams to include differential values of the coordinates (i.e. Cartesian, geodetic, ellipsoidal) may be consulted in [Soler, 1976].

6. CONCLUSIONS

With the introduction and proven reliability of modern geometric techniques such as interferometric observations to satellites, in particular the GPS constellation and Very Long Base Interferometry (VLBI) methods, new practical developments in both relative and absolute positioning have materialized. The capability of determining Cartesian coordinates in some prescribed system to precisions of a few parts in $1/10^8$ unquestionably opens a new range of oportunities for surveying, geodesy and geophysics, difficult to overstate.

The principal intent of this paper is to emphasize in a general but concise form the most important coordinate transformations that the new user of this data may face when applied to his/her particular analysis.

An obvious conclusion of the present exposition should be that the common although improper practice of designating local coordinate systems only by the direction of the axes (e.g. "east-north-up") does not suffice. It is imperative that the specific type of coordinate system be explicitly mentioned (e.g. spherical, geodetic, E- or W-longitude positive etc.). Only in this way may other required transformations such as the ones presented in the commutative diagram of Equ. (5.1) be properly applied.

Finally, another immediate consequence of this presentation is the possibility of rigorous propagation of errors by the use of the appropriate rotations and covariance transformations, in particular the determination of the rms in geodetic height ($\sigma_h \equiv \sigma_u$) at any point from the known covariance matrix Σ_{WGS72}. Undoubtedly, this may become an important neccessity in the near future when analysis of vertical deformations or relative geoidal undulations are attempted.

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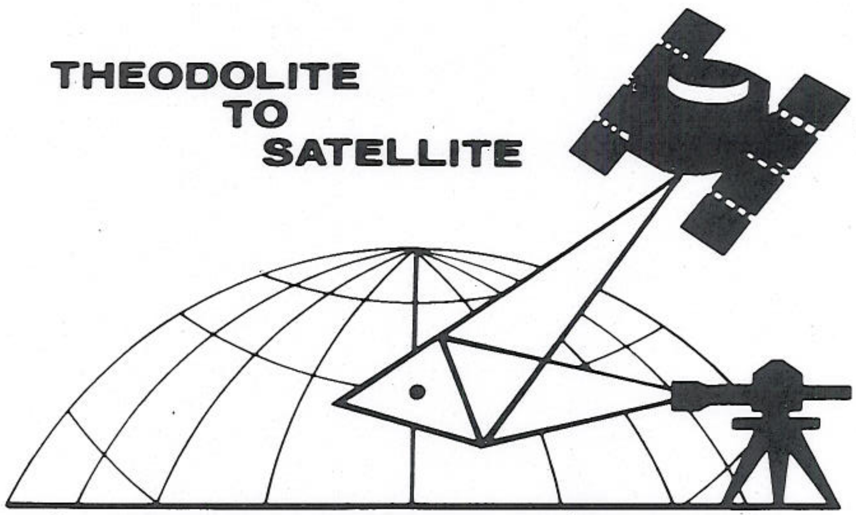

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